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GENERALLY COVARIANT FRESNEL EQUATION AND THE EMERGENCE OF THE LIGHT CONE STRUCTURE IN LINEAR PRE-METRIC ELECTRODYNAMICS

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We study the *propagation of electromagnetic waves* in a spacetime devoid of a metric but equipped with a *linear* electromagnetic spacetime relation $H \sim \chi \cdot F$. Here H is the electromagnetic excitation $(\mathcal{D}, \mathcal{H})$ and F the field strength (E, B), whereas χ (36 independent components) characterizes the electromagnetic permittivity/permeability of spacetime. We derive analytically the corresponding Fresnel equation and show that it is always quartic in the wave covectors. We study the 'Fresnel tensor density' $\mathcal{G}^{ijkl}$ as (cubic) function of χ and identify the leading part of χ (20 components) as indispensable for light propagation. Upon requiring electric/magnetic reciprocity of the spacetime relation, the leading part of χ induces the *light cone* structure of spacetime (9 components), i.e., the spacetime metric up to a function. The possible existence of an Abelian *axion* field (1 component of χ) and/or of a *skewon* field (15 components) and their effect on light propagation is discussed in some detail. The newly introduced skewon field is expected to be T-odd and related to dissipation. *file nonsym37.tex, 2002-03-24*

1. Introduction

In pre-metric electrodynamics, the axioms of electric charge and of magnetic flux conservation manifest themselves in the Maxwell equations for the excitation $H = (\mathcal{D}, \mathcal{H})$ and the field strength $F = (E, B)$, see [1]:

$$dH = J, \qquad dF = 0\,. \tag{1}$$

Here $J = (\rho, j)$ is the electric current. These equations keep their form in all reference frames and, since they are metric independent, in all gravitational fields as well. In order to develop this scheme into a predictive physical theory, one needs additionally a *spacetime relation* $H = H(F)$, which is a functional in general but is assumed here to be of a local nature. It should be well understood, this spacetime

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relation is a *universal* constitutive law for the *vacuum*. We do *not* treat the constitutive behavior of material media here, we rather search for a universal law for the vacuum, i.e., for the spacetime manifold itself [a].

The simplest assumption is that the spacetime relation is *local and linear*. If local coordinates x^i are given, with $i, j, ... = 0, 1, 2, 3$, we can decompose the excitation and field strength 2-forms into their components according to

$$H = \frac{1}{2}\, H_{ij}\, dx^i \wedge dx^j\,, \qquad F = \frac{1}{2}\, F_{ij}\, dx^i \wedge dx^j\,. \tag{2}$$

Then the *linear* spacetime relation, see [2,3], can be written as

$$H_{ij} = \frac{1}{4}\, \hat\epsilon_{ijkl}\, \chi^{klmn}\, F_{mn}\,. \tag{3}$$

Here $\hat\epsilon_{ijkl}$ is the Levi-Civita symbol, with $\hat\epsilon_{0123} = 1$. The quantity $\chi^{ijkl}(x)$ characterizes the electromagnetic properties of the vacuum and is as such of universal importance. It is an untwisted tensor density of weight $+1$ with 36 independent components. If we decompose it into irreducible pieces with respect to the 6-dimensional linear group, then we find

$$\chi^{ijkl} = {}^{(1)}\chi^{ijkl} + {}^{(2)}\chi^{ijkl} + {}^{(3)}\chi^{ijkl}\,, \quad \text{with} \quad 36 = 20 \oplus 15 \oplus 1 \tag{4}$$

independent components, respectively. The irreducible pieces of χ are defined as follows:

$$\begin{aligned} {}^{(2)}\chi^{ijkl} &:= \frac{1}{2}\left(\chi^{ijkl} - \chi^{klij}\right) = -{}^{(2)}\chi^{klij}\,, \quad {}^{(3)}\chi^{ijkl} := \chi^{[ijkl]}\,, \\ {}^{(1)}\chi^{ijkl} &:= \chi^{ijkl} - {}^{(2)}\chi^{ijkl} - {}^{(3)}\chi^{ijkl} = {}^{(1)}\chi^{klij}\,. \end{aligned} \tag{5}$$

Since no metric is available, we cannot form traces. The Abelian axion piece ${}^{(3)}\chi^{ijkl} =: \alpha(x)\,\epsilon^{ijkl}$ has been introduced by Ni [7,8]. Constitutive laws (for matter) with ${}^{(2)}\chi^{ijkl} \neq 0$ (non-vanishing "skewon fields" [b] have been discussed by Nieves and Pal [11,12]. They yield P- and CP-violating terms in the field equations. Thus all constitutive functions in (4) can claim respectability from a physical point of view in the framework of a linear response theory. We will come back to this question in Sec.3.

Our strategy will now be the following: After having specified the spacetime relation in (3), we have a predictive theory. Using Hadamard's method [13,14] for determining the propagation of waves in the spacetime specified so far, namely a 4-dimensional differentiable manifold carrying the tensor density χ^{ijkl} of (4), we will derive, in Sec.2, the *Fresnel equation* controlling the waves. It will turn out that the axion $\alpha(x)$ drops out and does not affect the waves, in spite of showing

[a] Nevertheless, since Maxwell's equations for material media have the same appearance as the vacuum equations (1), a constitutive relation for a material medium can be reminiscent of the *spacetime relation* — and we will profit from this analogy.

[b] The name 'skewon' has been used earlier in a post-Einsteinian theory of gravity by Mann, Moffat, and Taylor [9], see also Soleng and Eeg[10]. We are using this name in a different context since its old meaning doesn't seem to be in use any longer, see a title search in the data banks of SLAC and DESY.

up in the Maxwell equations. In other words, from χ^{ijkl} in (4), only ${}^{(1)}\chi^{ijkl}$, the leading piece, and ${}^{(2)}\chi^{ijkl}$, the skewon field, enter the Fresnel equation. We will give (for the first time) a generally covariant and explicit analytical derivation of the corresponding Fresnel equation and will show that its wave surfaces are in general of *quartic* order.

We will study in detail the conditions which must be fulfilled in order to factorize the quartic wave surfaces into two equal quadratic wave surfaces, the *light cone*. Up to a function, this amounts to a new *derivation of the metric of spacetime from electromagnetic data*.

In this article, we will restrict ourselves to *linear* pre-metric electrodynamics. Nevertheless, our methods and results are also useful in media with *non*linear relations between H and F, provided one studies electromagnetic *perturbations* on top of a background solution fulfilling the field equations. Then one can define, see Appendix A, an effective constitutive tensor density χ^{ijkl}_{eff} that is analogous to χ^{ijkl} of (4). In this way, our results, in particular our Fresnel equation, also apply to the work done in geometrical optics by De Lorenci, Novello, et al.[15] in nonlinear electrodynamics à la Born-Infeld, see also [16,17].

2. Wave propagation: Fresnel equation

We will study the propagation of a discontinuity of the electromagnetic field following the lines of Ref. [18]. The surface of discontinuity S is defined locally by a function Φ such that Φ = const. on S. Across S, the geometric Hadamard conditions are satisfied:

$$[F_{ij}] = 0, \qquad [\partial_i F_{jk}] = q_i\, f_{jk}, \tag{6}$$

$$[H_{ij}] = 0, \qquad [\partial_i H_{jk}] = q_i\, h_{jk}. \tag{7}$$

Here $[\mathcal{F}]\,(x)$ denotes the discontinuity of a function $\mathcal{F}$ across S, $q_i := \partial_i\Phi$ is the wave covector, and f_{ij} and h_{ij} are tensors describing the corresponding jumps of the derivatives of field strength and excitation. If we use Maxwell's vacuum equations $dH = 0$ and $dF = 0$, then (6) and (7) yield

$$\epsilon^{ijkl}\, q_j\, h_{kl} = 0\,, \qquad \epsilon^{ijkl}\, q_j\, f_{kl} = 0\,. \tag{8}$$

These equations admit non-vanishing wave covectors q_i, provided the constraints

$$\epsilon^{ijkl}\, f_{ij}\, f_{kl} = 0, \qquad \epsilon^{ijkl}\, h_{ij}\, h_{kl} = 0, \qquad \epsilon^{ijkl}\, f_{ij}\, h_{kl} = 0 \tag{9}$$

are fulfilled.

We assume that χ^{ijkl} is continuous across S. Then, we find from (3), (4) and (6), (7),

$$h_{ij} = \frac{1}{4}\,\hat{\epsilon}_{ijkl}\,\chi^{klmn}\, f_{mn}\,. \tag{10}$$

With this spacetime relation for the jumps, the conditions (8) can be rewritten as

$$\chi^{ijkl}\, q_j\, f_{kl} = \left({}^{(1)}\chi^{ijkl} + {}^{(2)}\chi^{ijkl}\right) q_j\, f_{kl} = 0\,, \qquad \epsilon^{ijkl}\, q_j\, f_{kl} = 0\,, \tag{11}$$

since ${}^{(3)}\chi^{ijkl}\, q_j\, f_{kl} = \alpha\, \epsilon^{ijkl}\, q_j\, f_{kl} = 0$. Thus the axion field α drops out even though it enters the Maxwell equations.

Technically, the system (11) was analyzed in Ref. [18] in the framework of a 3-(co)vector decomposition of the electromagnetic discontinuity tensor f_{ij}. Here we present a different, generally covariant derivation of the Fresnel equation by applying ideas of Tamm [19].

As a first step, eq.$(11)_2$ is solved by

$$f_{ij} = q_i a_j - q_j a_i\,. \tag{12}$$

The covector a_i is only defined up to a 'gauge' transformation

$$a_i \rightarrow a_i + \lambda q_i\,, \tag{13}$$

with an arbitrary scalar λ. We substitute (12) into $(11)_1$:

$$\chi^{ijkl}\, q_j q_k a_l = 0\,. \tag{14}$$

Because of (13), not all the equations in (14) are independent. In order to isolate the trivial parts of (14), it is convenient to pick a specific (anholonomic) *coframe* $\vartheta^\alpha = e_i{}^\alpha\, dx^i$, here $\alpha, \beta, \ldots = \hat 0, \hat 1, \hat 2, \hat 3$: Namely, we identify the zeroth leg of the coframe with the wave covector, that is, $\vartheta^{\hat 0} = q$ or, in components, $e_i{}^{\hat 0} = q_i$ or $q_\alpha = (1,0,0,0)$. Then

$$\chi^{\alpha\beta\gamma\delta}\, q_\beta q_\gamma a_\delta = \chi^{\alpha\hat 0\hat 0\delta}\, a_\delta = \chi^{\alpha\hat 0\hat 0 b}\, a_b = 0\,. \tag{15}$$

'Spatial' anholonomic indices run over $a, b, ... = \hat 1, \hat 2, \hat 3$. The zeroth component of (15) vanishes. Thus we find

$$\chi^{\hat 0 a\hat 0 b}\, a_b = W^{ab}\, a_b = 0\,, \quad \text{with} \quad W^{ab} := \chi^{\hat 0 a\hat 0 b}\,. \tag{16}$$

The necessary and sufficient condition for the existence of non-trivial solutions for a_b is the vanishing of the determinant of the 3×3 matrix W:

$$\mathcal{W} := \det W = \frac{1}{3!}\epsilon_{abc}\epsilon_{def} W^{ad} W^{be} W^{cf} = \frac{1}{3!}\epsilon_{abc}\epsilon_{def}\chi^{\hat 0 a\hat 0 d}\chi^{\hat 0 b\hat 0 e}\chi^{\hat 0 c\hat 0 f}\,. \tag{17}$$

This yields the Fresnel equation.

We can rewrite $\mathcal{W}$ in a fully 4-dimensional covariant manner. First, we observe that the 3-dimensional Levi-Civita symbol ϵ_{abc} is related to the 4-dimensional one by means of $\epsilon_{abc} \equiv \epsilon_{\hat 0 abc}$. Then we can extend one $\hat 0$-component of the constitutive tensors to a fourth summation index. As a result, we find the identity

$$\epsilon_{\hat 0 abc}\chi^{\hat 0 a\hat 0 d}\chi^{\hat 0 b\hat 0 e}\chi^{\hat 0 c\hat 0 f} = \epsilon_{\hat 0\beta\gamma\delta}\chi^{\hat 0\beta\hat 0 d}\chi^{\hat 0\gamma\hat 0 e}\chi^{\hat 0\delta\hat 0 f} = \frac{1}{2}\,\epsilon_{\alpha\beta\gamma\delta}\chi^{\alpha\beta\hat 0 d}\chi^{\hat 0\gamma\hat 0 e}\chi^{\hat 0\delta\hat 0 f}\,, \tag{18}$$

which holds true due to the (anti)symmetry properties of the Levi-Civita symbol and of the constitutive tensor. This allows us to rewrite (17) as

$$\mathcal{W} = \frac{1}{3!}\,\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\,\epsilon_{\hat 0 def}\,\chi^{\alpha\beta\hat 0 d}\chi^{\hat 0\gamma\hat 0 e}\chi^{\hat 0\delta\hat 0 f}\,. \tag{19}$$

Now we apply the same procedure to the second Levi-Civita symbol and finally obtain

$$\mathcal{W} = \frac{1}{4!}\epsilon_{\alpha\beta\gamma\delta}\,\epsilon_{\lambda\rho\sigma\tau}\chi^{\alpha\beta\hat{0}\rho}\chi^{\hat{0}\gamma\hat{0}\sigma}\chi^{\hat{0}\delta\lambda\tau}\,. \tag{20}$$

Since ${e_i}^{\hat{0}} = q_i$, this can, in coordinate components, be written as

$$\mathcal{W} = \frac{\theta^2}{4!}\,\epsilon_{mnpq}\,\epsilon_{rstu}\,\chi^{mnri}\,\chi^{jpsk}\,\chi^{lqtu}\,q_iq_jq_kq_l\,, \tag{21}$$

with $\theta := \det({e_i}^\alpha)$. We define the fourth order tensor density of weight $+1$

$$\mathcal{G}^{ijkl}(\chi) := \frac{1}{4!}\,\epsilon_{mnpq}\,\epsilon_{rstu}\,\chi^{mnr(i}\,\chi^{j|ps|k}\,\chi^{l)qtu}\,. \tag{22}$$

It is totally symmetric, $\mathcal{G}^{ijkl}(\chi) = \mathcal{G}^{(ijkl)}(\chi)$, and thus has 35 independent components (see [20]). With (22), we find the Fresnel equation in the generally covariant form

$$\mathcal{G}^{ijkl}(\chi)\,q_iq_jq_kq_l = 0\,. \tag{23}$$

The general Fresnel equation (23) is always a *quartic* equation in q_i despite the fact that it was derived from a determinant of a 3×3 matrix quadratic in the wave covectors. This corrects Denisov and Denisov [21] who claim that a particular case of the general linear constitutive law may yield a sixth order Fresnel equation; in [22], even a Fresnel equation of eighth order is claimed to hold.

The rest of our paper is devoted to finding conditions that will make the quartic Fresnel equation *factorize* into two quadratic ones and to determine under which circumstances the Fresnel equation turns out to be a perfect square.

For practical calculations, it is convenient to use a 1+3 coordinate decomposition similar to that of Refs. [23,24,18]. Correspondingly, we can rewrite our general result (23) as

$$\mathcal{W} = q_0^2\left(q_0^4 M + q_0^3 q_a\,M^a + q_0^2 q_a q_b\,M^{ab} + q_0 q_a q_b q_c\,M^{abc} + q_a q_b q_c q_d\,M^{abcd}\right) = 0\,, \tag{24}$$

with

$$M := \mathcal{G}^{0000}\,,\quad M^a := 4\mathcal{G}^{000a}\,,\quad M^{ab} := 6\mathcal{G}^{00ab}\,, \tag{25}$$

$$M^{abc} := 4\mathcal{G}^{0abc}\,,\quad M^{abcd} := \mathcal{G}^{abcd}\,. \tag{26}$$

In terms of the 3×3 matrices

$$\mathcal{A}^{ba} := \chi^{0a0b}\,,\qquad \mathcal{B}_{ba} := \frac{1}{4}\,\hat{\epsilon}_{acd}\,\chi^{cdef}\,\hat{\epsilon}_{efb}\,, \tag{27}$$

$$\mathcal{C}^b{}_a := \frac{1}{2}\,\hat{\epsilon}_{acd}\,\chi^{cd0b}\,,\qquad \mathcal{D}_b{}^a := \frac{1}{2}\,\chi^{0acd}\,\hat{\epsilon}_{cdb}\,, \tag{28}$$

the tensor density χ^{ijkl} can be written as a 6×6 matrix

$$\chi^{IK} = \begin{pmatrix} \mathcal{B}_{ab} & \mathcal{D}_a{}^b \\ \mathcal{C}^a{}_b & \mathcal{A}^{ab} \end{pmatrix}\,,\quad \text{with}\quad I,K,... = 1,2,...,6\,. \tag{29}$$

For the M's, we have explicitly:

$$
\begin{aligned}
M &= \det\mathcal{A}\,, && (30)\\
M^a &= -\hat\epsilon_{bcd}\left(\mathcal{A}^{ba}\,\mathcal{A}^{ce}\,\mathcal{C}^d_{\ e} + \mathcal{A}^{ab}\,\mathcal{A}^{ec}\,\mathcal{D}_e^{\ d}\right)\,, && (31)\\
M^{ab} &= \frac{1}{2}\,\mathcal{A}^{(ab)}\left[(\mathcal{C}^d{}_d)^2 + (\mathcal{D}_c{}^c)^2 - (\mathcal{C}^c{}_d + \mathcal{D}_d{}^c)(\mathcal{C}^d{}_c + \mathcal{D}_c{}^d)\right]\\
&\quad +(\mathcal{C}^d{}_c + \mathcal{D}_c{}^d)(\mathcal{A}^{c(a}\mathcal{C}^{b)}{}_d + \mathcal{D}_d{}^{(a}\mathcal{A}^{b)c}) - \mathcal{C}^d{}_d\mathcal{A}^{c(a}\mathcal{C}^{b)}{}_c\\
&\quad -\mathcal{D}_c{}^{(a}\mathcal{A}^{b)c}\mathcal{D}_d{}^d - \mathcal{A}^{dc}\mathcal{C}^{(a}{}_c\mathcal{D}_d{}^{b)} + \left(\mathcal{A}^{(ab)}\mathcal{A}^{dc} - \mathcal{A}^{d(a}\mathcal{A}^{b)c}\right)\mathcal{B}_{dc}\,, && (32)\\
M^{abc} &= \epsilon^{de(c|}\Big[\mathcal{B}_{df}(\mathcal{A}^{ab)}\,\mathcal{D}_e^{\ f} - \mathcal{D}_e^{\ a}\mathcal{A}^{b)f}\,) + \mathcal{B}_{fd}(\mathcal{A}^{ab)}\,\mathcal{C}^f_{\ e} - \mathcal{A}^{f|a}\mathcal{C}^{b)}_{\ e})\\
&\quad +\mathcal{C}^a_{\ f}\,\mathcal{D}_e^{\ b)}\,\mathcal{D}_d^{\ f} + \mathcal{D}_f^{\ a}\,\mathcal{C}^{b)}_{\ e}\,\mathcal{C}^f_{\ d}\Big]\,, && (33)\\
M^{abcd} &= \epsilon^{ef(c}\epsilon^{|gh|d}\,\mathcal{B}_{hf}\left[\frac{1}{2}\,\mathcal{A}^{ab)}\,\mathcal{B}_{ge} - \mathcal{C}^a_{\ e}\,\mathcal{D}_g^{\ b)}\right]. && (34)
\end{aligned}
$$

Here the M's, as totally symmetric tensors in 3 dimensions, carry $1\oplus 3\oplus 6\oplus 10\oplus 15 = 35$ independent components. All these results have been verified by using the computer algebra system Maple together with the tensor package GrTensor[c].

3. Axion and skewon

Before we draw explicitly conclusions from the Fresnel equation (23) or (24), it is instructive to study the structure of $\mathcal{G}^{ijkl}$. As we saw, this tensor density has 35 independent components, the same number as ${}^{(1)}\chi^{ijkl} + {}^{(2)}\chi^{ijkl}$. This can be understood as follows: Because of $(11)_1$, we have

$$\mathcal{G}^{ijkl}({}^{(3)}\chi) = 0\,. \tag{35}$$

This can also be directly verified by substituting ${}^{(3)}\chi \sim \epsilon$ into (22). Accordingly, if the spacetime relation (3) contained only the axion piece ${}^{(3)}\chi$, the propagation of electromagnetic waves would not be well-behaved. Therefore, in nature, it is excluded that χ consists merely of ${}^{(3)}\chi \sim \alpha$. On the other hand, if we substitute χ into $\mathcal{G}$, we find

$$\mathcal{G}^{ijkl}(\chi) = \mathcal{G}^{ijkl}({}^{(1)}\chi + {}^{(2)}\chi + {}^{(3)}\chi) = \mathcal{G}^{ijkl}({}^{(1)}\chi + {}^{(2)}\chi)\,. \tag{36}$$

Note that $\mathcal{G}$ depends on χ is a cubic way, that is, our last equation is by no means trivial. Consequently, if either ${}^{(1)}\chi$ or ${}^{(2)}\chi$ or both pieces exist in nature — and this is for sure — then the axion piece doesn't interfere with the local laws of the propagation of electromagnetic waves as determined by the Fresnel equation. Or expressed in a positive manner: The Abelian axion could be around, the laws of electrodynamics are compatible with it and the light propagation wouldn't be affected locally. There have been intensive experimental searches for axions, so far

[c]See http://grtensor.org .

without success [4,5,6]. However, the axion remains a serious candidate for a particle search in experimental high energy physics.

The axion piece has one further interesting property. The kinetic electromagnetic energy-momentum current in pre-metric electrodynamics reads, see [26,3],

$$^{\mathrm{k}}\Sigma_\alpha := \frac{1}{2}\left[F\wedge(e_\alpha\rfloor H) - H\wedge(e_\alpha\rfloor F)\right]. \tag{37}$$

If we substitute the spacetime relation (3) into it, we find, in reminiscence of (36),

$$^{\mathrm{k}}\Sigma_\alpha(\chi) = {}^{\mathrm{k}}\Sigma_\alpha\left({}^{(1)}\chi + {}^{(2)}\chi + {}^{(3)}\chi\right) = {}^{\mathrm{k}}\Sigma_\alpha\left({}^{(1)}\chi + {}^{(2)}\chi\right). \tag{38}$$

Moreover, $^{\mathrm{k}}\Sigma_\alpha({}^{(3)}\chi) = 0$. Clearly then, the axion does not contribute to the electromagnetic energy-momentum current. Nevertheless, as Ni [7,8] has shown, it is possible to develop a reasonable theory for the Abelian axion. Accordingly, the axion piece $^{(3)}\chi$ of the constitutive tensor density χ cannot be dismissed a priori.

Let's turn then to the skewon piece $^{(2)}\chi$. The name we derived from the skew- or antisymmetric 6×6 matrix $^{(2)}\chi$ can be mapped to. Commonly, this piece is not considered seriously. The conventional argument runs as follows, see Post [2]. Suppose a Lagrangian 4-form L exists for the electromagnetic field. In general $H \sim \partial L/\partial F$. If H is assumed to be linear in F, as is done in (3), then L reads $L \sim H\wedge F \sim \chi\cdot F\wedge F = {}^{(1)}\chi\cdot F\wedge F + {}^{(3)}\chi\cdot F\wedge F$, since the piece with $^{(2)}\chi$ drops out of L because of the antisymmetry $^{(2)}\chi^{ijkl} = -\,{}^{(2)}\chi^{klij}$, see (5). Hence,

$$L(\chi) = L({}^{(1)}\chi + {}^{(3)}\chi) = L({}^{(1)}\chi) + L({}^{(3)}\chi). \tag{39}$$

The term with $^{(1)}\chi$ eventually becomes the Maxwell Lagrangian, the term with $^{(3)}\chi$ part of the axion Lagrangian. Since we conventionally assume that all information of a physical system is coded into its Lagrangian, we reject $^{(2)}\chi^{ijkl} \neq 0$ as being unphysical.

However, as we saw already in (38), if a piece drops out from a certain expression, it does not necessarily imply that this piece lost its right of existence. Eq.(39) only shows that L is 'insensitive' to the skewon piece. In contrast, the electromagnetic energy-momentum current (37) definitely 'feels' the contribution from the skewon,

$$^{\mathrm{k}}\Sigma_\alpha({}^{(2)}\chi) \neq 0, \tag{40}$$

that is, the skewon piece does carry electromagnetic energy-momentum. Thereby, it could displays its presence. Clearly then, we expect that $^{(2)}\chi$ influences light propagation. And, indeed, it does. By inspection, we find

$$\mathcal{G}^{ijkl}({}^{(1)}\chi + {}^{(2)}\chi) = \mathcal{G}^{ijkl}({}^{(1)}\chi) + \overline{\mathcal{G}}^{ijkl}({}^{(1)}\chi, {}^{(2)}\chi), \tag{41}$$

with

$$\overline{\mathcal{G}}^{ijkl}({}^{(1)}\chi, {}^{(2)}\chi) \neq 0, \tag{42}$$

even though

$$\mathcal{G}^{ijkl}({}^{(2)}\chi) = 0. \tag{43}$$

In other words, like the axion, the skewon cannot propagate light decently unless the ${}^{(1)}\chi$ piece participates. Moreover,

$$\mathcal{G}^{ijkl}({}^{(2)}\chi + {}^{(3)}\chi) = 0\,, \tag{44}$$

i.e., the skewon and the axion piece cannot exist alone or together, the 'leading' ${}^{(1)}\chi$ piece is indispensable. If we knew more about the detailed structure of $\overline{\mathcal{G}}^{ijkl}({}^{(1)}\chi, {}^{(2)}\chi)$, then we could better judge which modes of the skewon are compatible with present-day experiments.

In any case, in linear pre-metric electrodynamics a Lagrangian is not assumed to exist a priori. Therefore, it is not alarming that ${}^{(2)}\chi$ drops out from the proto-Lagrangian L. Since it carries electromagnetic energy-momentum, we will take *the possible existence of* ${}^{(2)}\chi^{ijkl}$ *for granted.*

What is then the possible physical meaning of ${}^{(2)}\chi$? This has already been discussed by Nieves and Pal [11,12]. But let us first collect some formalism. Consider a certain vector field $\xi = \xi^\alpha e_\alpha$, with the basis e_α of the tangent vector space at each point of spacetime. Then we can transvect the energy-momentum current (37) with ξ^α:

$$\mathcal{Q} := \xi^\alpha\, {}^{\rm k}\Sigma_\alpha = \frac{1}{2}\left[F \wedge (\xi\rfloor H) - H \wedge (\xi\rfloor F)\right]\,. \tag{45}$$

The scalar-valued 3-form $\mathcal{Q}$ is expected to be related to conserved quantities provided we can find suitable (Killing type) vector fields ξ. Therefore we determine its exterior derivative and find after some algebra,

$$d\mathcal{Q} = (\xi\rfloor F) \wedge J + \frac{1}{2}\left(F \wedge \mathcal{L}_\xi H - H \wedge \mathcal{L}_\xi F\right)\,, \tag{46}$$

or, in holonomic components, with $\mathcal{Q}^i := \epsilon^{ijkl} Q_{jkl}/6$, $\mathcal{J}^i := \epsilon^{ijkl} J_{jkl}/6$, and $\mathcal{H}^{ij} := \epsilon^{ijkl} H_{kl}/2$,

$$\partial_i \mathcal{Q}^i = \xi^k F_{kl}\, \mathcal{J}^l + \frac{1}{4}\left(F_{kl}\, \mathcal{L}_\xi \mathcal{H}^{kl} - \mathcal{H}^{kl}\, \mathcal{L}_\xi F_{kl}\right)\,. \tag{47}$$

Here $\mathcal{L}_\xi$ denotes the Lie derivative along the vector ξ. Now we substitute the linear spacetime relation (3), or $\mathcal{H}^{kl} = \chi^{klmn} F_{mn}/2$, and find

$$\partial_i \mathcal{Q}^i = \xi^k F_{kl}\, \mathcal{J}^l + \frac{1}{8}\left[F_{kl}\, \mathcal{L}_\xi(\chi^{klmn} F_{mn}) - \chi^{klmn} F_{mn}\, \mathcal{L}_\xi F_{kl}\right]\,. \tag{48}$$

We apply the Leibniz rule of the Lie derivative and rearrange a bit:

$$\partial_i \mathcal{Q}^i = \xi^k F_{kl}\, \mathcal{J}^l + \frac{1}{8}\left[(\mathcal{L}_\xi \chi^{ijkl})\, F_{ij} F_{kl} + (\chi^{ijkl} - \chi^{klij})\, F_{ij}\, \mathcal{L}_\xi F_{kl}\right]\,. \tag{49}$$

We substitute the irreducible pieces of χ^{ijkl}. Then we have

$$\partial_i \mathcal{Q}^i = \xi^k F_{kl}\, \mathcal{J}^l + \frac{1}{8}\, \mathcal{L}_\xi \left({}^{(1)}\chi^{ijkl} + {}^{(3)}\chi^{ijkl}\right) F_{ij} F_{kl} + \frac{1}{4}\, {}^{(2)}\chi^{ijkl}\, F_{ij}\, \mathcal{L}_\xi F_{kl}\,. \tag{50}$$

Here it is manifest that ${}^{(1)}\chi$ and the axion ${}^{(3)}\chi$ behave qualitatively different as compared to the skewon ${}^{(2)}\chi$. If ${}^{(1)}\chi$ and ${}^{(3)}\chi$ carry a symmetry along ξ such that $\mathcal{L}_\xi {}^{(1)}\chi^{ijkl} = 0$ and $\mathcal{L}_\xi {}^{(3)}\chi^{ijkl} = 0$, then, in vacuum, i.e., for $\mathcal{J}^i = 0$, we still have

non-conservation because of the offending term ${}^{(2)}\chi\, F\dot{F}$. Here the dot symbolizes the Lie derivative along ξ. If ξ can be interpreted as a 'time' direction, then $\mathcal{Q}$ represents the electromagnetic energy density.

In any case we see that ${}^{(2)}\chi$ induces a *dissipative* term with a first 'time' derivative. This is what we might have expected since, in general, dissipative phenomena cannot be described conveniently in a Lagrangian framework. It is then our hypothesis that the skewon piece ${}^{(2)}\chi$ can represent a field which is *odd under T transformations*. This is also what had been discussed by Nieves and Pal [11,12]. Of course, we must investigate how this *skewon*, as we may call it in a preliminary way, disturbs the light cone and whether there is perhaps only a viable subclass of the 15 independent components of the skewon. An isotropic skewon s could be constructed by putting in (29) $\mathcal{C}^a{}_b = s\,\delta^a_b$ and $\mathcal{D}_a{}^b = -s\,\delta^b_a$, see Nieves and Pal [12].

Let us try then to collect our results in order to get a rough picture of the physical meaning of these different irreducible pieces. The piece ${}^{(1)}\chi^{ijkl}$ is indispensable for an appropriate propagation of light, as we saw in (35), (43), and (44). Thus we call it the leading piece of χ^{ijkl}.

If ${}^{(1)}\chi^{ijkl}$ exists alone and for the spacetime relation (3) electric/magnetic reciprocity is assumed to hold additionally, see Sec.4, then, up to an unknown function, the metric of spacetime can be derived including its Lorentzian signature [23,24,25]. One can think of this reduction in the way that electric/magnetic reciprocity cuts the 20 components of ${}^{(1)}\chi^{ijkl}$ into half, that is, only 10 components are left for the metric. Modulo an undetermined function, we have then 9 remaining components. These 9 remaining components of ${}^{(1)}\chi^{ijkl}$ determine the *light cone* at each point of spacetime. Accordingly, in ${}^{(1)}\chi^{ijkl}$ the light cone of spacetime is hidden and thereby conventional Maxwell-Lorentzian vacuum electrodynamics as well. To put it more geometrically, the first irreducible piece ${}^{(1)}\chi$, via electric/magnetic reciprocity, yields the *conformal* structure of spacetime. In this sense, there is no doubt that ${}^{(1)}\chi^{ijkl}$ is the principal part of the constitutive tensor density χ of the vacuum. But, as we argued above, there are no a priori reasons for excluding either the axion ${}^{(3)}\chi$ or the skewon ${}^{(2)}\chi$. In particular, the Fresnel equation and the electromagnetic energy-momentum current can accommodate both pieces in a plausible way.

4. Almost complex structure

Let us now search for a condition restricting the constitutive tensor density χ^{ijkl} of the spacetime relation (3). Since the times of Maxwell and Heaviside, in the equations of electrodynamics a certain symmetry was noticed between the electric and the magnetic quantities and was used in theoretical discussions. We formulate electric/magnetic reciprocity as follows [3]: Given the energy-momentum current (37). It is electric-magnetic reciprocal, i.e., it remains invariant ${}^{\mathrm{k}}\Sigma_\alpha \to {}^{\mathrm{k}}\Sigma_\alpha$ under the transformation

$$H \to \zeta F\,, \quad F \to -\frac{1}{\zeta}\,H\,, \tag{51}$$

with the dimensionful pseudo-scalar function $\zeta = \zeta(x)$.

We now postulate *electric/magnetic reciprocity of the spacetime relation (3).* Under (51), the spacetime relation transforms into

$$\zeta\, F_{ij} = -\frac{1}{4\,\zeta}\,\hat\epsilon_{ijkl}\,\chi^{klmn}\,H_{mn}\,. \tag{52}$$

If we substitute this into (3), we find, after some algebra, the consistency condition

$$-\,\frac{1}{8\,\zeta^2}\,(\hat\epsilon_{ijmn}\,\chi^{mnpq})\,(\hat\epsilon_{pqrs}\,\chi^{rskl}) = \delta^{kl}_{ij}\,, \tag{53}$$

with the function

$$\zeta^2 := -\frac{1}{96}\,(\hat\epsilon_{ijmn}\,\chi^{mnpq})\,(\hat\epsilon_{pqrs}\,\chi^{rsij})\,. \tag{54}$$

In order to allow for a more compact notation, we introduce the dimensionless density $\overset{\circ}{\chi}{}^{ijkl} := \chi^{ijkl}/\zeta$ and define

$$\kappa_{ij}{}^{kl} = \frac{1}{2}\,\hat\epsilon_{ijmn}\,\overset{\circ}{\chi}{}^{mnkl}\,. \tag{55}$$

Then the *closure relation* reads

$$\kappa_{ij}{}^{mn}\kappa_{mn}{}^{kl} = -2\delta^{kl}_{ij} \tag{56}$$

or, even more compactly, as 6×6 matrix equation,

$$\boldsymbol{\kappa}\,\boldsymbol{\kappa} = -\,\mathbf{1}_6. \tag{57}$$

Mathematically this means that the operator $\boldsymbol{\kappa}$ represents an almost complex structure on the space of 2-forms. For the special case of ${}^{(2)}\chi = {}^{(3)}\chi \equiv 0$, such closure relations were first discussed by Toupin [27], Schönberg [28], and Jadczyk [29].

Let us now make the closure relation explicit. We turn back to the constitutive 6×6 matrix (29). We define dimensionless 3×3 matrices $\overset{\circ}{\mathcal{A}} := \mathcal{A}/\zeta$, etc. In terms of these dimensionless matrices (we immediately drop the small circle for convenience), the closure relation reads,

$$\mathcal{A}^{ac}\mathcal{B}_{cb} + \mathcal{C}^a{}_c\mathcal{C}^c{}_b = -\delta^a_b, \tag{58}$$

$$\mathcal{C}^a{}_c\mathcal{A}^{cb} + \mathcal{A}^{ac}\mathcal{D}_c{}^b = 0, \tag{59}$$

$$\mathcal{B}_{ac}\mathcal{C}^c{}_b + \mathcal{D}_a{}^c\mathcal{B}_{cb} = 0, \tag{60}$$

$$\mathcal{B}_{ac}\mathcal{A}^{cb} + \mathcal{D}_a{}^c\mathcal{D}_c{}^b = -\delta^b_a. \tag{61}$$

Assume $\det\mathcal{B} \neq 0$. Then we can find the general non-degenerate solution. We define the matrix K_{ab} by

$$K := \mathcal{B}\mathcal{C}\,, \qquad \text{i.e.} \qquad \mathcal{C} = \mathcal{B}^{-1}K\,, \tag{62}$$

and substitute it into (60),

$$\mathcal{D} = -K\mathcal{B}^{-1}. \tag{63}$$

Next, we solve (58) with respect to $\mathcal{A}$:

$$\mathcal{A} = -\mathcal{B}^{-1} - \mathcal{B}^{-1}K\mathcal{B}^{-1}K\mathcal{B}^{-1}. \tag{64}$$

We multiply (64) by $\mathcal{C}$ from the left and by $\mathcal{D}$ from the right, respectively, and find with (62) and (63),

$$\mathcal{C}\mathcal{A} = -\mathcal{B}^{-1}K\mathcal{B}^{-1} - \mathcal{B}^{-1}K\mathcal{B}^{-1}K\mathcal{B}^{-1}K\mathcal{B}^{-1}, \tag{65}$$

$$\mathcal{A}\mathcal{D} = +\mathcal{B}^{-1}K\mathcal{B}^{-1} + \mathcal{B}^{-1}K\mathcal{B}^{-1}K\mathcal{B}^{-1}K\mathcal{B}^{-1}. \tag{66}$$

Thus, (59) is automatically satisfied. Accordingly, only (61) has still to be checked. We compute its first and second term of its left side,

$$\mathcal{B}\mathcal{A} = -1 - K\mathcal{B}^{-1}K\mathcal{B}^{-1}\,, \tag{67}$$

$$\mathcal{D}^2 = K\mathcal{B}^{-1}K\mathcal{B}^{-1}\,, \tag{68}$$

and find that it is fulfilled, indeed.

Summing up, we have derived the general solution of the closure relation (57) in terms of two arbitrary matrices $\mathcal{B}$ and K as

$$\mathcal{A} = -\mathcal{B}^{-1} - \mathcal{B}^{-1}K\mathcal{B}^{-1}K\mathcal{B}^{-1}\,, \tag{69}$$

$$\mathcal{C} = \mathcal{B}^{-1}K\,, \tag{70}$$

$$\mathcal{D} = -K\mathcal{B}^{-1}\,. \tag{71}$$

The solution thus has $2 \times 9 = 18$ independent components. Alternatively, one can write the solution as

$$\mathcal{A} = -(1 + \mathcal{C}^2)\mathcal{B}^{-1}\,, \tag{72}$$

$$\mathcal{D} = -\mathcal{B}\mathcal{C}\mathcal{B}^{-1}\,, \tag{73}$$

which is parametrized by the arbitrary matrices $\mathcal{B}$ and $\mathcal{C}$ with altogether 18 independent components.

5. Spacetime metric

Can we construct the spacetime metric by using the new general solution for the generalized closure relation? Technically, this is equivalent to the question: Can the general Fresnel equation be reduced to the light-cone structure? Here we study the conditions for such a reduction.

Let us decompose the arbitrary matrix $\mathcal{B}$ into its symmetric and antisymmetric parts,

$$\mathcal{B}_{ab} = b_{ab} + \hat{\epsilon}_{abc}n^c\,, \quad \text{with} \quad b_{ab} := \mathcal{B}_{(ab)}\,, \quad n^c := \epsilon^{cab}\,B_{[ab]}\,. \tag{74}$$

Note that b_{ab} contributes to ${}^{(1)}\chi$ and n^c to ${}^{(2)}\chi$. Now we can lower the index of n^a by means of b_{ab}, namely, $n_a := b_{bc}\,n^c$ and $n^2 := n^c n_c = b_{ab}\,n^a n^b$, and, provided $\det b \neq 0$, we can raise an index by b^{ab} which denotes the inverse of b_{ab}. We find $\det \mathcal{B} = \det b + n^2$, and the inverse of (74) reads

$$\mathcal{B}^{ab} = \frac{1}{\det b + n^2}\left(\bar{b}^{ab} + n^a n^b - \epsilon^{abc}\,n_c\right). \tag{75}$$

Here the symmetric matrix $\bar{b}^{ab}$ is the matrix of the minors of b_{ab}. If $\det b \neq 0$, then $\bar{b}^{ab} = b^{ab} \det b$.

Let us find out what is qualitatively new in the asymmetric case as compared to the previously studied symmetric case [18]. For this purpose we consider the particular solution of the closure relation for $K = 0$. Consequently, $\mathcal{C} = \mathcal{D} = 0$ and $\mathcal{A} = -\mathcal{B}^{-1}$. Then, by substituting (74) into (24), we obtain:

$$\mathcal{W} = -\frac{q_0^2}{(\det b + n^2)}\left[q_0^4 - 2q_0^2\left(q^2 \det b - (qn)^2\right) + \left(q^2 \det b + (qn)^2\right)^2\right]. \tag{76}$$

Here we used the abbreviation $(qn) := q_a\, n^a$.

In general, this expression is neither a square of a quadratic polynomial nor a product of two quadratic polynomials. In other words, neither a light cone nor a birefringence (double light cone) structure arises generically. In order to study the reduction conditions, let us assume that the Fresnel equation is a product of two quadratic equations for q_i, i.e., the spacetime 'medium' is birefringent. Accordingly, for (76) we make the general ansatz

$$\mathcal{W} = -\frac{q_0^2}{(\det b + n^2)}(q_0^2 - \alpha)(q_0^2 - \beta) = -\frac{q_0^2}{(\det b + n^2)}\left[q_0^4 - (\alpha + \beta)q_0^2 + \alpha\beta\right], \tag{77}$$

with some polynomials α and β of order 2 in q_a. This implies the relations

$$\alpha + \beta = 2\left(\bar{q}^2 - (qn)^2\right), \qquad \alpha\beta = \left(\bar{q}^2 + (qn)^2\right)^2, \tag{78}$$

with $\bar{q}^2 := q_a q_b \bar{b}^{ab}$. Since α and β enter symmetrically in (78), the solutions of this nonlinear system can be given in the form

$$\alpha = -\left[(qn) + \sqrt{-\bar{q}^2}\right]^2, \qquad \beta = -\left[(qn) - \sqrt{-\bar{q}^2}\right]^2. \tag{79}$$

Thus, the question of the reducibility of the Fresnel equation translates into the algebraic problem of whether the square root $\sqrt{-\bar{q}^2}$ is a real linear polynomial in q_a. There are three cases, depending on the rank of the 3×3 matrix b_{ab}.

(i) When b_{ab} has rank 3, in other words, when $\det b \neq 0$, then we can write $\bar{q}^2 = q_a q_b\, b^{ab} \det b$, and the general conclusion is that *no factorization into light-cones* is possible (the roots α are complex), unless $n^a = 0$. This latter condition implies that the constitutive tensor is symmetric, and the results of [23,18,25] are recovered.

(ii) When b_{ab} has rank 2, i.e., $\det b = 0$, but at least one of the minors is nontrivial. Then, without loss of generality, we can assume the following structure of the matrix b:

$$b_{ab} = \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{12} & b_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{80}$$

Its only non-vanishing minor is

$$\bar{b}^{33} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}b_{22} - b_{12}^2 \neq 0. \tag{81}$$

Note that (80) is the most general form of a rank 2 matrix b_{ab}, up to a renaming of the coordinates. In order to avoid complex solutions, we have to assume that the minor $\overline{b}^{33} = -\,\mu^2 < 0$, so that $\sqrt{-\overline{q}^2} = \mu q_3$. Then (79) leads to

$$\alpha = -\,\left[q_1 n^1 + q_2 n^2 + q_3(n^3 + \mu)\right]^2, \qquad \beta = -\,\left[q_1 n^1 + q_2 n^2 + q_3(n^3 - \mu)\right]^2 . \tag{82}$$

The interpretation is clear: we have birefringence, i.e., two light-cones. In this case, the Fresnel equation is found to be

$$\begin{aligned}\mathcal{W} = -\,\frac{q_0^2}{b_{11}(n^1)^2 + 2b_{12}n^1n^2 + b_{22}(n^2)^2}\ & \left(q_0^2 + \left[q_1 n^1 + q_2 n^2 + q_3(n^3 + \mu)\right]^2\right) \\ & \times\left(q_0^2 + \left[q_1 n^1 + q_2 n^2 + q_3(n^3 - \mu)\right]^2\right) = 0 .\end{aligned} \tag{83}$$

Then we can read off, up to conformal factors, the components of the two corresponding 'metric' tensors defining the light-cones:

$$g_1^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (n^1)^2 & n^1n^2 & n^1(n^3+\mu) \\ 0 & n^1n^2 & (n^2)^2 & n^2n^3 \\ 0 & n^1(n^3+\mu) & n^2n^3 & (n^3+\mu)^2 \end{pmatrix}, \tag{84}$$

$$g_2^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (n^1)^2 & n^1n^2 & n^1(n^3-\mu) \\ 0 & n^1n^2 & (n^2)^2 & n^2n^3 \\ 0 & n^1(n^3-\mu) & n^2n^3 & (n^3-\mu)^2 \end{pmatrix}. \tag{85}$$

We can verify that $\det(g_1^{ij}) = \det(g_2^{ij}) = (n^1)^2(n^2)^2\overline{b}^{33} = -(n^1)^2(n^2)^2\mu^2 < 0$, so that both metrics have the correct Lorentzian signature.

(iii) When the 3×3 matrix b_{ab} has rank 1: In this case all the minors are zero, i.e., $\overline{q}^{ab} = 0$, which corresponds to the case 2 for $\mu = 0$. We then see that the Fresnel equation reduces to a single light cone, but the resulting metric is degenerated, since $\det(g^{ij}) = 0$.

6. Discussion and conclusion

In this paper we extended appreciably our earlier results [23,24,18,25] of linear premetric electrodynamics by relaxing the symmetry and the closure property of the constitutive tensor χ^{ijkl} of spacetime. In the most general case of a linear spacetime relation (when both *closure* and *symmetry* are absent), the wave propagation is governed by the newly derived Fresnel type equation (23) which is still of quartic order, contrary to the claims in Ref. [21]. We have deduced the tensor density $\mathcal{G}^{ijkl}$ in (22) which induces the generally covariant form of the Fresnel equation (23). We studied some properties of $\mathcal{G}^{ijkl}$ and showed that ${}^{(1)}\chi^{ijkl}$, as the leading part of χ^{ijkl}, is indispensable for a decent propagation of electromagnetic waves. However, also each of the two remaining parts, the axion ${}^{(3)}\chi^{ijkl}$ and the skewon ${}^{(2)}\chi^{ijkl}$, has a well-defined physical meaning. The axion drops out from light propagation.

High energy physicist search for such type of particles. The skewon seems to be T-odd and related to dissipative processes. Its influence on light propagation, see (42), deserves further study.

In Sec.4, the general solution of the closure relation is presented for the case of an asymmetric linear χ^{ijkl}. Within a particular class of the asymmetric solutions of the closure relation, the reduction of the Fresnel equation was analyzed in detail. We have demonstrated that some of these solutions can yield birefringence. A preliminary study of the general case with $K \neq 0$ shows that the conclusions remain qualitatively the same as those presented in Sec.5.

Thus, we can conclude that the conditions of closure and symmetry of χ^{ijkl} are sufficient for the existence of a well-defined light cone structure. If any of these conditions is violated, the light cone structure seems to be lost. The necessary conditions have still to be found. Of course, if one wants to study possible violations of Lorentz invariance by means of the skewon ${}^{(2)}\chi^{ijkl}$, e.g., then the light cone structure cannot be considered as sacrosanct any longer.

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Appendix A. An effective constitutive tensor

Consider an arbitrary local spacetime relation $H = H(F)$. In components, the corresponding Maxwell equation, for $J = 0$, reads

$$\epsilon^{ijkl}\,\partial_j\,H_{kl} = 0\,. \tag{A.1}$$

A small perturbation ΔF of the electromagnetic field around some background $\bar{F}$ can be written as $F = \bar{F} + \Delta F$. Then, to first order in the perturbation, we have for the excitation

$$H_{kl}(F) = H_{kl}(\bar{F}) + \frac{1}{2}\left.\frac{\partial H_{kl}}{\partial F_{mn}}\right|_{\bar{F}} \Delta F_{mn}\,. \tag{A.2}$$

Inserting (A.2) into (A.1) and assuming that the background field $\bar{F}$ is a solution of (A.1), i.e. $\epsilon^{ijkl}\,\partial_j\,H_{kl}(\bar{F}) = 0$, we obtain an equation for the perturbation:

$$\partial_j\left(\chi^{ijkl}_{\rm eff}\,\Delta F_{kl}\right) = 0\,, \qquad \chi^{ijkl}_{\rm eff} := \frac{1}{2}\epsilon^{ijmn}\left.\frac{\partial H_{mn}}{\partial F_{kl}}\right|_{\bar{F}}\,. \tag{A.3}$$

The effective constitutive tensor density $\chi^{ijkl}_{\rm eff}$ will, in general, depend on the local constitutive law and on the background field $\bar{F}$.

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